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Rank 2 Commuting Ordinary Differential Operators and Darboux Conjugates of KdV

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Abstract—By considering the factorizations (flags) and associated (simultaneous) second order Darboux transformations of the square and cube of an arbitrary second order Schrödinger operator, we generate commuting ordinary differential operators of orders four and six with a singular elliptic spectrum. This procedure generates true rank 2 commutative algebras. Under the KdV flow, each such factorization (flag) leads to an integrable equation for which the corresponding Darboux transformation generates a Lax-type operator as one of a commuting pair of orders four and six with singular elliptic spectrum. Hence, these integrable equations are Darboux conjugates of KdV.

Keywords—Darboux/Bäcklund transformations, KdV equation, Commuting ordinary differential operators.

1. INTRODUCTION

The theories of commuting ordinary differential operators (ODO's) and integrable equations are naturally linked by the representation of these equations, such as KdV, in Lax form [1]. In this context, commutativity of ODO's is a reflection of stationarity with respect to particular integrable flows within a hierarchy. Part of understanding this link is the interpretation of Darboux (Bäcklund) transformations of ODO's for the associated integrable flows. However, commuting ODO's enjoy a rich and classical theory [1–4] quite apart from their association through Lax operators with integrable equations.

We touch upon both aspects of commutativity for the special case of Darboux conjugates of the square of a Schrödinger (KdV Lax) operator $K = D^2 + (1/2)V_0$. By considering the factorizations (flags) of K^2 and applying second order Darboux transformations, we generate, from the trivial commuting pair (K^2, K^3), nontrivial (and nonself-adjoint) commuting pairs or orders four and six with the same singular joint spectrum $\mu^2 = \lambda^3$. As an application of this, any K whose centralizer $\mathcal{C}(K)$ contains no odd-order ODO's (i.e., is rank 2) generates, by Darboux conjugation, a nontrivial rank 2 algebra with this singular spectrum. This procedure is made explicit by using expressions, derived in an elementary way, for the potential V_0 in terms of a wronskian of eigenfunctions in the factorizing flag. Under the KdV flow, these expressions lead naturally to integrable equations for this wronskian of the flag of K^2. These equations, which include the rank 2 singular Krichever-Novikov (s-KN) [5–10] and Ur-KdV equations [11], have Darboux conjugates of K^2 as their Lax type operators. Hence, our Darboux construction provides an alternative framework by which to associate these equations with KdV as compared to the symmetry [5,12]

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX and Painlevé [13,14] approaches. Further, this association is useful in generating explicit solutions of these equations as can be seen by considering the vanishing rational solutions [15] of KdV where everything is explicitly calculable (cf. [8]).

2. DARBOUX TRANSFORMATIONS AND COMMUTING ODO'S

In this section, we examine the result of Darboux conjugation of K^2 by one of its second order factors as determined by three interesting flags.

For a fixed eigenvalue κ^2, consider the general Schrödinger operator

$$K = D^2 + \frac{1}{2}V - \kappa^2, \tag{1}$$

where $D = \frac{d}{dx}$ and $V = V(x)$. We will often write $V_0 = V - 2\kappa^2$ to incorporate the eigenvalue as a translation of V. Let ϕ and $\tilde{\phi}$ span ker K and have $W(\phi, \tilde{\phi}) = \phi\tilde{\phi}' - \phi'\tilde{\phi} = \rho$, a nonzero constant. Further (cf. [5]), let ψ and $\tilde{\psi}$ be any eigenfunctions which complete a basis of ker K^2 and satisfy $K\psi = \phi$ and $K\tilde{\psi} = \tilde{\phi}$. The operator K^2 can now be factored into the product of first order factors in several different ways by composing an appropriate flag from these eigenfunctions. We write, for the flag $\mathcal{F}$: $\{0\} \subseteq \{\chi_0\} \subseteq \{\chi_0, \chi_1\} \subseteq \{\chi_0, \chi_1, \chi_2\} \subseteq \{\chi_0, \chi_1, \chi_2, \chi_3\}$, with the χ_i's taken from the set $\{\phi, \tilde{\phi}, \psi, \tilde{\psi}\}$; $K^2 = (D + \nu_3)(D + \nu_2) \cdot (D + \nu_1)(D + \nu_0)$, where $\nu_i = \partial_x \ln(W_i/W_{i+1})$, and W_i, $i = 0, 1, 2, 3$, is the wronskian of the first i eigenfunctions of the flag with the convention that $W_0 = 1$ [1, Proposition 4.10]. Throughout this paper, and independent of the particular flag under consideration, we will abbreviate $W_2 = W(\chi_0, \chi_1) = w$. Define the two corresponding second order factors of K^2 to be

$$\begin{aligned} P &= (D + \nu_3)(D + \nu_2) = D^2 - q_1 D + p_0, \\ Q &= (D + \nu_1)(D + \nu_0) = D^2 + q_1 D + q_0, \end{aligned} \tag{2}$$

so that $K^2 = P \cdot Q$. The operator $K^3 = KP \cdot Q$ trivially commutes with K^2 and the commuting pair (K^2, K^3) has joint (eigenvalue) spectrum $\mu^2 = \lambda^3$. Because of the form of (1), and since, from (2), $q_1 = \nu_0 + \nu_1$, we deduce that $q_1 = -w'/w$ while q_0 satisfies the first order linear ODE

$$2q_0' - 2q_1 q_0 + q_1'' - 3q_1 q_1' + q_1^3 + q_1 V_0 - V_0' = 0, \quad \text{and} \quad p_0 = V_0 - q_0 - 2q_1' + q_1^2. \tag{3}$$

Thus, for known V_0, (3) gives a closed system for determining the factorization $K^2 = P \cdot Q$.

REMARK 1. Once an explicit expression for V_0 in terms of w is available, it follows that P and Q can be determined solely in terms of this wronskian.

The Darboux transformation studied here is that of conjugation by Q, which we write as

$$\begin{aligned} L &= \widetilde{K^2} = Q \cdot P = \left(D^2 + \frac{1}{2}\tilde{c}_2\right)^2 + 2\tilde{c}_1 D + \tilde{c}_1' + \tilde{c}_0, \\ M &= \widetilde{K^3} = Q \cdot KP, \end{aligned} \tag{4}$$

for some as yet unknown coefficients $\tilde{c}_i$. Note that, in general, we cannot expect L to be a perfect square, or even (formally) self-adjoint (i.e., $\tilde{c}_1 \equiv 0$). It is clear from their definitions that $[L, M] = 0$ and the commuting pair (L, M) again has joint spectrum $\mu^2 = \lambda^3$. Moreover, the rank of the algebra $\mathbb{C}[L, M]$, defined to be the gcd of the orders of all its elements, equals that of $\mathbb{C}[K^2, K^3]$ (cf. [7]). The main aim of this section is to determine the explicit formulae for the coefficients of L (and M).

The first two eigenfunctions in the flag $\mathcal{F}$ determine $w = W(\chi_0, \chi_1)$. There are therefore only three flags of interest in terms of the Darboux transformation (4). These are the flags $\mathcal{F}_i$, $i = 1, 2, 3$, whose second subspace is

$$\mathcal{F}_1 : \{\phi, \psi\} \quad \text{with} \quad w = W(\phi, \psi), \tag{5i}$$

$$\mathcal{F}_2 : \{\phi, \tilde{\psi}\} \quad \text{with} \quad w = W(\phi, \tilde{\psi}), \tag{5ii}$$

$$\mathcal{F}_3 : \{\psi, \tilde{\psi}\} \quad \text{with} \quad w = W(\psi, \tilde{\psi}). \tag{5iii}$$

The remaining flags with second subspaces $\{\phi,\tilde\phi\}$, $\{\tilde\phi,\tilde\psi\}$ and $\{\tilde\phi,\psi\}$ produce, respectively, the factorization $K^2 = K\cdot K$ ($P=Q=K$) and the same results as $\mathcal{F}_1$ and $\mathcal{F}_2$.

With Remark 1 in mind, we begin by establishing some expressions for V in terms of the respective wronskians w.

THEOREM 1. *Let $K = D^2+(1/2)V-\kappa^2$ have zero eigenfunctions ϕ and $\tilde\phi$ for which $W(\phi,\tilde\phi)=\rho$ is nonzero (constant) and assume that ψ and $\tilde\psi$ satisfy $K\psi=\phi$ and $K\tilde\psi=\tilde\phi$. For the flags $\mathcal{F}_i$, $i=1,2,3$, in (5), the following expressions for V hold,*

(i) $\mathcal{F}_1$: $V = 2\kappa^2 - w'''/w' + (1/2)w''^2/w'^2$, *where* $w=W(\phi,\psi)$;

(ii) $\mathcal{F}_2$: $V = 2\kappa^2 - w'''/w' + (1/2)w''^2/w'^2 - (1/2)\rho^2/w'^2$, *where* $w=W(\phi,\tilde\psi)$;

(iii) $\mathcal{F}_3$: $V = 2\kappa^2 - w'''/w' + (1/2)w''^2/w'^2 + 2(\rho w - c^2)/w'^2$, *where* $w=W(\psi,\tilde\psi)$ *and c is a constant.*

PROOF. We prove only (ii) to illustrate the method. From $\tilde\psi'' = -(1/2)V_0\tilde\psi + \tilde\phi$ and $\phi'' = -(1/2)V_0\phi$, we obtain $w' = \phi\tilde\phi$, $w'' = \phi\tilde\phi' + \phi'\tilde\phi$ and $w''' = -V_0w' + 2H$, where $H = \phi'\tilde\phi'$. Now $H' = -(1/2)V_0w''$ and so H satisfies the first order linear ODE $w'H' + w''H = (1/2)w''w'''$. Solving for H gives $H = w''^2/(4w') - c^2/w'$, where c^2 is an integration constant, and substituting into the expression for w''' gives (ii). We defer for the moment the proof that $c^2=\rho^2/4$ (see Proposition 1 (ii) below). Similar manipulations prove (i) and (iii) (proved in [8]), except that for (i), no ODE arises. ∎

The constants in Theorem 1 can be determined by computing explicit expressions for ψ and $\tilde\psi$ from the factorization $K = (D+\phi'/\phi)(D-\phi'/\phi)$. Since $W(\phi,\tilde\phi)=\rho$, we must have $\tilde\phi = \rho\phi\int\phi^{-2} + b\phi$ for some constant b. Hence, solving for ψ and $\tilde\psi$ we obtain that

$$\psi = \phi\int\left[\phi^{-2}\int\phi^2\right] + B_1\phi + B_2\phi\int\phi^{-2}, \quad \text{and}$$
$$\tilde\psi = \rho\phi\int\left[\phi^{-2}\int\left(\phi^2\int\phi^{-2}\right)\right] + b\phi\int\left[\phi^{-2}\int\phi^2\right] + \tilde B_1\phi + \tilde B_2\phi\int\phi^{-2}, \tag{6}$$

for arbitrary integration constants B_1, B_2, $\tilde B_1$ and $\tilde B_2$, must be the general expressions for these eigenfunctions (it is understood that the integrals are performed without additive constants).

PROPOSITION 1. *With the notation in (6), the constants in Theorem 1 are given by:*

(ii) $c^2=\rho^2/4$ (*c^2 is the constant in the proof of Theorem 1 (ii)), and*

(iii) $c^2 = (\tilde B_2 - bB_2 + \rho B_1 + \rho k)^2/4$ *where* $k = \int\phi^{-2}\int\phi^2 - (\int\phi^{-2})(\int\phi^2) + \int\phi^2\int\phi^{-2}$.

PROOF. This is a straightforward computation accomplished by first using (6) to obtain w and then substituting in the respective cases of Theorem 1 with $V_0 = -2\phi''/\phi$ to obtain c^2. ∎

REMARK 2. Note that for given ϕ and $\tilde\phi$, an appropriate choice of the constants B_i and $\tilde B_i$ in (6) can always be made to ensure that c in (iii) will vanish. In light of (ii), we will in this case, as in Theorem 1 (ii), make the choice $c=\rho/2$ throughout.

The results of Theorem 1 can now be used to solve the ODE in (3) and so determine the explicit factors of $K^2 = P\cdot Q$. Since P is determined by V_0, q_1 and q_0 (see (3)), and expressions for V_0 have been obtained in Theorem 1, we need only list the coefficients of Q.

PROPOSITION 2. *The operator $K^2 = (D^2+(1/2)V_0-\kappa^2)^2$ has the factorization $K^2 = P\cdot Q$ where $Q = D^2+q_1D+q_0$ is, in each case, given by $q_1 = -w'/w$, and q_0 is, respectively,*

(i) $\mathcal{F}_1$: $q_0 = -(1/2)w'''/w' + (1/4)w''^2/w'^2 + (1/2)w''/w = (1/2)V_0 + (1/2)w''/w$, *where* $w=W(\phi,\psi)$;

(ii) $\mathcal{F}_2$: $q_0 = -(1/2)w'''/w' + (1/4)w''^2/w'^2 - (1/4)\rho^2/w'^2 + (1/2)w''/w + (1/2)\rho/w = (1/2)V_0 + (1/2)w''/w + (1/2)\rho/w$, *where* $w=W(\phi,\tilde\psi)$;

(iii) $\mathcal{F}_3$: $q_0 = -(1/2)w'''/w' + (1/4)w''^2/w'^2 + (\rho w - c^2)/w'^2 + (1/2)w''/w + c/w = (1/2)V_0 + (1/2)w''/w + c/w$, *where* $w=W(\psi,\tilde\psi)$.

Knowing both factors P and Q, we can now compute explicitly the coefficients of the fourth order operator $L = Q \cdot P$ which is the result of the second order Darboux transformation that is conjugation by Q (cf. (4)).

THEOREM 2. *Let $L = Q \cdot P = (D^2 + (1/2)\tilde{c}_2)^2 + 2\tilde{c}_1 D + \tilde{c}_1' + \tilde{c}_0$ be the result of (Darboux) conjugation of $K^2 = P \cdot Q$ by Q. Then for the flags $\mathcal{F}_i$, $i = 1, 2, 3$, in (5), the coefficients of L are, in each case, $\tilde{c}_2 = V_0 + 4\partial_x^2 \ln w$, where $V_0 = V - 2\kappa^2$ is given by the corresponding case in Theorem 1, and*

(i) *for $\mathcal{F}_1$: $\tilde{c}_1 = 0$ and $\tilde{c}_0 = 0$;*
(ii) *for $\mathcal{F}_2$: $\tilde{c}_1 = \rho w'/w^2$ and $\tilde{c}_0 = -\rho^2/w^2$;*
(iii) *for $\mathcal{F}_3$: $\tilde{c}_1 = 2cw'/w^2$ and $\tilde{c}_0 = 2\rho/w - 4c^2/w^2$.*

COROLLARY 1. *Let $M = L_+^{3/2}$ where the plus subscript denotes the differential operator part and L is given by any of the cases in Theorem 2 with $w(x)$ an arbitrary (nonconstant) function and ρ and c arbitrary constants. Then $[L, M] = 0$ and the commuting pair (L, M) has spectrum $\mu^2 = \lambda^3$. Moreover, the rank of the algebra $\mathbb{C}[L, M]$ is equal to that of the centralizer $\mathcal{C}(K)$.*

REMARK 3. The case $c = 0$ in Theorem 2 (iii) was encountered in [8]. Then, L is formally self-adjoint and, under the KdV flow, plays the role of a Lax type operator for the s-KN equation (see Section 3 below). The cases in Theorem 2 can be easily identified as particular (singular) instances of the commuting ODO's in [2,3].

As was hinted earlier, one of the important applications of results like Corollary 1 is to generate examples of rank two algebras by considering a suitable Schrödinger operator K. We continue the example, started in [8], associated with the Lamé operators.

EXAMPLE 1. Let $\kappa^2 = 0$ and $V = -2a(a+1)x^{-2}$, i.e., $K = D^2 - a(a+1)x^{-2}$, where a is a constant. By taking $\phi = x^{-a}$ and $\tilde{\phi} = x^{1+a}$, so that $\rho = 1 + 2a$, and $\tilde{\psi} = (1/2)x^{3+a}/(3 + 2a)$, we compute that $w = W(\phi, \tilde{\psi}) = x^2/2$. Hence, by Theorem 2 (ii) and Corollary 1, the operators L and $M = L_+^{3/2}$ with coefficients $\tilde{c}_2 = -2(4 + a(a + 1))x^{-2}$, $\tilde{c}_1 = 4(1 + 2a)x^{-3}$ and $\tilde{c}_0 = -4(1 + 2a)^2 x^{-4}$, commute, have spectrum $\mu^2 = \lambda^3$ and generate a rank 2 algebra $\mathbb{C}[L, M]$ whenever a is not an integer.

3. SECOND ORDER DARBOUX CONJUGATES OF KdV

We now examine the consequences of the KdV flow for the results established in the previous section and introduce the new parameterization $v = -\rho/w$ which produces simpler and better recognized forms.

Suppose now that $K_0 = D^2 + (1/2)V$ evolves under the KdV flow; namely, $\partial_t K_0 = [(K_0^{3/2})_+, K_0]$, which in turn implies the KdV equation for V:

$$V_t = \frac{3}{4} V V_x + \frac{1}{4} V_{xxx}. \tag{7}$$

The KdV equation leads to corresponding evolution equations for the wronskians w (and v) while the corresponding L's are operators of Lax type for these evolution equations (in v).

LEMMA 2. *Let V be given by any of the expressions in Theorem 1 with ρ, c and κ^2 independent of t. Then $V_t = Mw_t$, $V_x = Mw_x$ and $(3/4)VV_x + (1/4)V_{xxx} = MF(w)$ where, correspondingly, the ODO's M and the expressions $F(w)$ are given by*

(i) $M = -w_x^{-1} D(D - w_{xx}/w_x)D$, $F(w) = (1/4)(w_{xxx} - (3/2)w_{xx}^2/w_x) + (3/2)\kappa^2 w_x$;
(ii) $M = -w_x^{-1}(D + \rho/w_x)(D - w_{xx}/w_x - \rho/w_x)D$, $F(w) = (1/4)(w_{xxx} - (3/2)w_{xx}^2/w_x) + (3/8)\rho^2/w_x^2 + (3/2)\kappa^2 w_x$;
(iii) $M = -w_x^{-1} D(D - w_{xx}/w_x)D + 4(c^2 - \rho w)/w_x^3 D + 2\rho/w_x^2$, $F(w) = (1/4)(w_{xxx} - (3/2) w_{xx}^2/w_x) - (3/2)(\rho w - c^2)/w_x + (3/2)\kappa^2 w_x$.

PROOF. These facts can be verified by direct calculation or, alternatively, by solving the differential equations $MF(w) = (3/4)VV_x + (1/4)V_{xxx}$ for $F(w)$. ∎

COROLLARY 2. *With the assumptions of Lemma 2, then modulo ker M, the KdV equation for V implies the following evolution equations for the corresponding wronskians w:*

(i) $\mathcal{F}_1$: $w_t/w_x = (1/4)\{w;x\} + (3/2)\kappa^2$;

(ii) $\mathcal{F}_2$: $w_t/w_x = (1/4)\{w;x\} + (3/8)\rho^2/w_x^2 + (3/2)\kappa^2$;

(iii) $\mathcal{F}_3$: $w_t/w_x = (1/4)\{w;x\} + (3/2)(c^2 - \rho w)/w_x^2 + (3/2)\kappa^2$;

where $\{w;x\}$ is the (partial) Schwarzian derivative. Conversely, any solution w of any of these evolution equations generates a KdV solution V via the corresponding formula in Theorem 1.

We now change parameters to $v = -\rho/w$ in order to see the Darboux conjugates L, of K^2, assume the mantle of Lax type operators for the resulting transformed equations from Corollary 2. This change produces simpler expressions for these L's.

THEOREM 3. *Let $L = (D^2 + (1/2)\tilde{c}_2)^2 + 2\tilde{c}_1 D + \tilde{c}_1' + \tilde{c}_0$; then the Lax representation*

$$\partial_t L = \left[L_+^{3/4}, L\right] \tag{8}$$

leads (for the listed $\tilde{c}_i$) to the following integrable equations conjugate to the KdV equation:

(i) $\tilde{c}_2 = -v_{xxx}/v_x + (1/2)v_{xx}^2/v_x^2$, $\tilde{c}_1 = \tilde{c}_0 = 0$, *leading to*

$$\frac{v_t}{v_x} = \frac{1}{4}\{v;x\} + \frac{3}{2}\kappa^2;$$

(ii) $\tilde{c}_2 = -v_{xxx}/v_x + (1/2)v_{xx}^2/v_x^2 - (1/2)v^4/v_x^2$, $\tilde{c}_1 = v_x$, $\tilde{c}_0 = -v^2$, *leading to*

$$\frac{v_t}{v_x} = \frac{1}{4}\{v;x\} + \frac{3}{8}\frac{v^4}{v_x^2} + \frac{3}{2}\kappa^2;$$

(iii) $\tilde{c}_2 = -v_{xxx}/v_x + (1/2)v_{xx}^2/v_x^2 - 2(v^3 + s^2v^4)/v_x^2$, $\tilde{c}_1 = 2sv_x$, $\tilde{c}_0 = -2v - 4s^2v^2$, *where s $(= c/\rho)$ is a constant, leading to*

$$\frac{v_t}{v_x} = \frac{1}{4}\{v;x\} + \frac{3}{2}\frac{v^3(1+s^2v)}{v_x^2} + \frac{3}{2}\kappa^2.$$

Moreover, in each case, L and $M = L_+^{3/2}$ commute, have joint spectrum $\mu^2 = \lambda^3$, and L is a second order Darboux conjugate of the operator $K^2 = (D^2 + (1/2)V_0)^2$.

PROOF. This is simply a restatement of the results so far after the substitution $v = -\rho/w$ in Lemma 2, and Corollary 2. The Lax form (8) is just the KdV flow imposed on the Darboux conjugate L of K^2. As in Lemma 2, similar operators $\tilde{M}$ arise and lead to the given evolution equations for v. ∎

The equation in (i) is the Ur-KdV equation [11] which may also be obtained using Darboux in a different way to that considered here. The equation in (ii) is, after the substitutions $u = 1/v$, $\lambda_1 = 3/2$ and rescaling $t \to 4t$, (4.1.16) in [12]. The equation in (iii) is the s-KN equation [7–9], and can be placed in a better recognized form by the transformation $v \to v/(1 - s^2v)$. The operator L in this case can be further Darboux-transformed to produce KP solutions (see [7,8]).

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